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OF UNIFORM ACCURACY**

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TOTAL VARIATION DIMINISHING (TVD) SCHEMES OF UNIFORM ACCURACY

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Abstract

Explicit second-order accurate finite-difference schemes for the approximation of hyperbolic conservation laws are presented. These schemes are nonlinear even for the constant coefficient case. They are based on first-order accurate upwind schemes. Their accuracy is enhanced by locally replacing the first-order one-sided differences with either second-order one-sided differences or central differences or a blend thereof. The appropriate local difference stencils are selected such that they give TVD schemes of uniform second-order accuracy in the scalar, or linear systems, case. Like conventional TVD schemes, the new schemes avoid a Gibbs phenomenon at discontinuities of the solution, but they do not switch back to first-order accuracy, in the sense of truncation error, at extrema of the solution. The performance of the new schemes is demonstrated in several numerical tests.

INTRODUCTION

The notion of TVD goes back to Harten's ground breaking paper [2]. He set out to construct second-order accurate finite-difference schemes for hyperbolic conservation laws which do not exhibit the spurious oscillations near discontinuities of the solution, as generated by the more classical second-order methods, such as Lax-Wendroff [7]. Harten discovered that any TVD scheme is a monotonicity preserving scheme which always produces oscillation-free weak solutions of hyperbolic problems. As Van Leer did in his early works [14,15], Harten realized that monotonicity preserving schemes can be only extended to higher-order accuracy when they are nonlinear even for the constant coefficient case. Following Roe [9] and Sweby [12], such nonlinear schemes can be interpreted as first-order accurate upwind schemes whose excessive numerical dissipation is counterbalanced by adding just enough antidiffusion to gain resolution while still suppressing pre- and post-shock oscillations. The amount of antidiffusion is monitored by flux limiters. Yet, at extrema of the solution, all formulations suggested so far for those limiters [1,9,12,14,15] restore the baseline first-order scheme by setting the antidiffusive flux to zero. This perpetual damping of extrema reduces the global error of TVD schemes to be of only first order in the L_∞ norm [4], which impedes particularly the approximate solutions for transient wave-tracking problems.

To remedy this shortcoming, Harten and Osher [4] recently introduced their "Uniformly high-order accurate NONoscillatory (UNO) schemes." When applied to nonoscillatory initial conditions, UNO schemes preserve their nonoscillatory nature although they are allowed to accentuate local extrema.

Thus, they are not only no longer TVD, but they also do no longer ensure that the numerical solution converges to the physically correct one [2,4,12]. The question whether one really has to sacrifice the TVD property for the sake of uniform accuracy motivated the present study.

This paper describes explicit high resolution schemes of uniform second-order accuracy which are also TVD in the scalar case. The next section briefly recapitulates the Riemann problem and Roe's [8] (first-order accurate) approximate Riemann solver to set the stage for the construction of those high resolution schemes. As usually done, first the rather watered-down special case of scalar conservation laws is examined which gives us a class of new TVD schemes of Uniform second-order Accuracy (in short: TVDUA schemes). The design principles of those TVDUA schemes are then applied to hyperbolic systems using a field-by-field decomposition. Eventually, the TVDUA schemes and their high-resolution counterparts for hyperbolic systems (which are not necessarily TVD for nonlinear systems) compete with conventional TVD schemes in several comparative numerical studies, where they show their superior accuracy at virtually no computational extra cost.

THE PROBLEM AND ITS FIRST-ORDER ACCURATE APPROXIMATE SOLUTIONS

Consider an initial value problem for a hyperbolic system of conservation laws

$$Q_t + F(Q)_x = 0 \quad (2.1a)$$

with

$$Q(x, t=0) = Q_0(x) \quad (2.1b)$$

where the initial data $Q_0(x)$ are assumed to be piecewise smooth functions which are either periodic or of compact support. The state vector $Q = (q_1, q_2, \dots, q_m, \dots, q_M)^T$ is a column vector of M unknowns and the flux $F(Q)$ is a vector-valued function of M components. The system in (2.1a) is hyperbolic in the sense that the $M \times M$ Jacobian $A(Q) = \partial F / \partial Q$ has M real eigenvalues λ_m . The relation between the eigenvalues λ_m and the Jacobian $A(Q)$ is given by

$$A(Q) = \sum_{m=1}^M r_m \lambda_m \ell_m \quad (2.2)$$

where the columns r_m are linearly independent right eigenvectors and the rows ℓ_m give an orthonormal set of left eigenvectors (i.e., $\sum_i r_i \ell_j = \delta_{ij}$).

Let Q_j and Q_{j+1} denote two piecewise constant states of a Riemann problem. These two states are separated from each other by one or more wave families, which can be a shock, an expansion fan, or a contact discontinuity. Each wave family $m(=1, 2, \dots, M)$ is propagated with its own wave speed which is given by the corresponding eigenvalue. In order to apply this wave model approach (cf. [16]) to equation (2.1a), consider (2.1a) for a computational cell with its centroid at x_j and its two interfaces at $x_{j \pm 1/2}$. The flux $F_{j+1/2}$ is defined by

$$F_{j+1/2} = F_j - (\sum_m \lambda_m^- \alpha_m)_{j+1/2} = F_{j+1} - (\sum_m \lambda_m^+ \alpha_m)_{j+1/2} \quad (2.3)$$

where $\alpha_{m, j+1/2} = \ell_{m, j+1/2} \Delta_{j+1/2} Q$ with $\Delta_{j+1/2}(\cdot) = (\cdot)_{j+1} - (\cdot)_j$ defines the strength of the m -th wave and $\lambda_m^\pm = (|\lambda_m| \pm \lambda_m)/2$.

Following Roe [8], a mean value matrix

$$\bar{A}(Q_j, Q_{j+1}) = (\bar{A}^+ - \bar{A}^-)_{j+1/2} = (\bar{\lambda}_m^+ \bar{\lambda}_m^+ - \bar{\lambda}_m^- \bar{\lambda}_m^-)_{j+1/2}$$

with the properties

$$(i) \quad \bar{A}(Q, Q) = A(Q)$$

$$(ii) \quad \bar{A}(Q_j, Q_{j+1}) \Delta_{j+1/2} Q = \Delta_{j+1/2} F$$

is utilized to write (2.3) as

$$F_{j+1/2} = F_j - \bar{A}_{j+1/2}^- \Delta_{j+1/2} Q = F_{j+1} - \bar{A}_{j+1/2}^+ \Delta_{j+1/2} Q. \quad (2.4)$$

Using (2.4), a first-order accurate finite-difference approximation to (2.1) is easily constructed

$$\Delta Q_j^n - (\tau/h) (\bar{A}_{j+1/2}^- \Delta_{j+1/2} Q - \bar{A}_{j-1/2}^+ \Delta_{j-1/2} Q)^n = 0 \quad (2.5)$$

where $\Delta()^n = ()^{n+1} - ()^n$, τ is the time step size, and $h = x_{j+1/2} - x_{j-1/2}$. An approximate solution computed with (2.5) tends to the exact solution for small data due to property (i). Property (ii) ensures that the approximate solution is exact if Q_j and Q_{j+1} are connected by a single discontinuity, regardless of its size. This becomes evident by comparing (ii) with the jump condition

$$v \Delta_{j+1/2} Q = \Delta_{j+1/2} F$$

where V , the discontinuity speed, is an eigenvalue of $\bar{A}(Q_j, Q_{j+1})$ and $\Delta_{j+1/2} Q$ is the corresponding eigenvector.

Remark 2.1: The derivation of the scheme in (2.5) tacitly implies that we have to deal with a Riemann problem at each and every interface. Then, the assumption of piecewise constant states Q_j is equivalent to replacing the actual function $Q(x)$ in $x_{j-1/2} < x < x_{j+1/2}$ with its cell average

$$Q_j = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} Q \, dx.$$

Consequently, the scheme in (2.5) and its higher-order variants as discussed in this paper give approximations to cell averages of the exact solution rather than to its point values at the centroids x_j .

Remark 2.2: The scheme in (2.5) is not entropy satisfying [6] and admits stationary expansion shocks as its steady solution. Although this can be readily rectified [2,3], these entropy considerations will be ignored here for the sake of a simplified presentation.

UNIFORMLY SECOND-ORDER ACCURATE TVD SCHEMES FOR SCALAR CONSERVATION LAWS

For clarity we first consider the simplest possible case of a linear scalar equation

$$q_t + a q_x = 0, \quad a > 0. \quad (3.1)$$

The assumption $a > 0$ does not impede generality; the contrary case is easily treated by symmetry. Applying the scheme in (2.5) to (3.1) yields

$$\Delta q_j^n + v \Delta_{j-1/2} q^n = 0, \quad v = a(\tau/h) \quad (3.2)$$

where h is restricted to be constant, although results for irregular grids follow in a similar manner. A general second-order formulation of (3.2) reads

$$\Delta q_j^n + v[1+0.5(1-v)(\phi_j^- - \phi_{j-1}^+)]\Delta_{j-1/2} q^n = 0. \quad (3.3)$$

As usual [5,9,10,12,14], the quantities ϕ_j^\bullet are defined as functions of the ratio of two consecutive gradients

$$r_j^\pm = \begin{cases} (\Delta_{j-1/2} q^n / \Delta_{j+1/2} q^n)^{\pm 1} & \text{for } \Delta_{j\pm 1/2} q^n \neq 0 \\ 0 & \text{for } \Delta_{j\pm 1/2} q^n = 0 \end{cases} \quad (3.4)$$

The scheme in (3.3) is conservative when the quantities ϕ_j^\pm satisfy Sweby's symmetry property [12] (i.e., $\phi_j^- = r_j^- \phi_j^+$). We will christen the quantities ϕ_j^\pm with stencil selectors since they determine the underlying difference stencil of a particular finite-difference scheme. Choose $\phi_j^- = r_j^-$ and $\phi_j^+ = \text{const} = 1$, and (3.3) recovers the second-order Lax-Wendroff scheme [7]. For $\phi_j^- = \text{const} = 1$ and $\phi_j^+ = r_j^+$, (3.3) is identical with the second-order Warming-Beam scheme [17]. Both linear schemes exhibit spurious oscillations at discontinuities of $q(x)$ since they are not TVD [12] (i.e., not monotonicity preserving [2]). We turn now to find stencil selectors $\phi(r_j^\pm)$ which make (3.3) both TVD and uniformly second-order accurate.

For a general scheme written in the form

$$\Delta q_j^n - (c_{j+1/2}^- \Delta_{j+1/2} q - c_{j-1/2}^+ \Delta_{j-1/2} q)^n = 0 \quad (3.5)$$

where $c_{j\mp 1/2}^\pm$ are data dependent (i.e., functions of the set $\{q_n^n\} = \{\dots, q_{j-1}^n, q_j^n, q_{j+1}^n, \dots\}$), it is easily shown [2] that sufficient conditions for it to be TVD are the inequalities

$$0 \leq c_{j+1/2}^\pm \quad \text{and} \quad c_{j+1/2}^+ + c_{j+1/2}^- \leq 1. \quad (3.6)$$

A comparison of (3.3) and (3.5) yields

$$c_{j+1/2}^- = 0 \quad \text{and} \quad c_{j+1/2}^+ = v[1 + 0.5(1-v)(\phi_{j+1}^- - \phi_j^+)]. \quad (3.7)$$

The scheme in (3.3) is TVD if

$$-2/(1-v) \leq \phi_{j+1}^- - \phi_j^+ \leq 2/v, \quad 0 < v < 1 \quad (3.8)$$

An infinite multitude of functions $\phi_j^\pm = \phi(r_j^\pm)$ satisfy condition (3.8). To derive one that is useful for the construction of uniformly second-order TVD schemes, additional constraints are required. Two of those constraints are

$$\phi(0) = 0 \quad (3.9a)$$

$$\phi(1) = 1. \quad (3.9b)$$

Condition (3.9a) is necessary to make (3.3) TVD [2], and condition (3.9b) immediately follows from Sweby's symmetry condition (i.e., $\phi_j^+ = r_j^+ \phi_j^-$), which makes scheme (3.3) conservative.

Having earlier established the notion of stencil selectors, makes it quite straightforward to define further bounds on ϕ_j^\pm . Second-order central differences do not introduce any dissipation and they have less dispersion than fully one-sided second-order differences. This suggests to control ϕ_j^\pm such that the scheme in (3.3) recovers the Lax-Wendroff scheme whenever there is no conflict with the TVD conditions in (3.8) and (3.9a). A function $\phi(r_j^\pm)$ that satisfies these conditions is an antisymmetric extension of the min mod limiter

$$\phi(r_j^\pm) = \max(\gamma, \min(r_j^\pm, 1)), \quad \gamma = -1 \quad (3.10)$$

which, for $\gamma=0$, is the conventional min mod limiter [2,4,5,9,12,18]. The conventional min mod limiter, like various other TVD limiters [9,12], returns $\phi(r_j^\pm < 0) = 0$ which reduces the scheme in (3.3) to a first-order upwind scheme even at smooth extrema of the solution $q(x)$. Applying the antisymmetric min mod limiter in (3.10) to the scheme in (3.3) shows that second-order accuracy is maintained even across extrema (i.e., for $r_j^\pm < 0$). Since the scheme becomes TVD of uniform accuracy, we call it a TVDUA scheme. It is interesting to note that equation (3.10) admits negative values for $\phi(r_j^\pm)$; a phenomenon incompatible with the notion of "antidiffusion" [9,12], which is often used to devise second-order TVD schemes.

Remark 3.1: In the remainder of this paper we will speak of first-order TVD schemes when $\phi(r_j^\pm) = \text{const} = 0$, second-order TVD schemes when $\phi(r_j^\pm)$ is defined by a conventional min mod limiter, and of TVDUA schemes when $\phi(r_j^\pm)$ is defined by an extension of the conventional TVD limiter as in (3.10).

The extension of the TVDUA scheme to a nonlinear conservation law like

$$q_t + a(q)q_x = 0 \quad (3.11)$$

is straightforward. Applying the scheme in (2.5) to (3.11), we obtain

$$\Delta q_j^n - (v_{j+1/2}^- \Delta_{j+1/2} q - v_{j-1/2}^+ \Delta_{j-1/2} q)^n = 0 \quad (3.12)$$

where $v_{j\mp 1/2}^\pm = a_{j\mp 1/2}^\pm \tau/h$ is now a local CFL number with $a^\bullet = (|a| \pm a)/2$. Using the general form in (3.5), a conservative second-order extension to (3.12) which accounts for the local flow direction has the coefficients

$$c_{j+1/2}^- = v_{j+1/2}^- [1 - 0.5(1 - v_{j+1/2}^-) \phi_{j+1}^-] + 0.5 v_{j-1/2}^- (1 - v_{j-1/2}^-) \phi_j^+ \quad (3.13a)$$

$$c_{j+1/2}^+ = v_{j+1/2}^+ [1 - 0.5(1 - v_{j+1/2}^+) \phi_j^+] + 0.5 v_{j+3/2}^+ (1 - v_{j+3/2}^+) \phi_{j+1}^- \quad (3.13b)$$

The coefficients in (3.13) preserve conservancy whenever the limiters ϕ_j^\pm satisfy Sweby's symmetry property [12] (i.e., $\phi_j^- = r_j^- \phi_j^+$), and they satisfy the TVD conditions (3.6) when

$$-2/v_{j+1/2}^- < \phi_{j+1}^- - \phi_j^+ < 2/v_{j+1/2}^+ \quad (3.14a)$$

$$-2/(1 - v_{j+1/2}^+) < \phi_{j+1}^- - \phi_j^+ < 2/(1 - v_{j+1/2}^-) \quad (3.14b)$$

These restraints of ϕ_j^\bullet are satisfied for all $0 < v_{j+1/2}^\pm < 1$ when ϕ_j^+ is chosen is already defined in equation (3.10).

UNIFORMLY SECOND-ORDER ACCURATE HIGH RESOLUTION SCHEMES FOR HYPERBOLIC SYSTEMS

The TVDUA schemes are formally extended to hyperbolic systems in the usual field by field decomposition [1,2,4,5,9,12,18]. Defining characteristic variables:

$$w_m = \ell_m Q$$

and assuming the constant coefficient case (i.e., A is a constant $M \times M$ matrix), equation (2.1a) is written as a set of M decoupled linear scalar conservation laws

$$(w_m)_t + \lambda_m (w_m)_x = 0 \quad (4.1)$$

A first-order upwind approximation to (4.1) reads

$$\Delta w_j^n - (\tau/h) (\lambda_m^- \Delta_{j+1/2} w_m - \lambda_m^+ \Delta_{j-1/2} w_m)^n = 0 \quad (4.2)$$

The uniformly second-order accurate TVDUA scheme for linear scalar conservation laws as defined in (3.3) and (3.10) is straightforwardly applicable to (4.2):

$$\begin{aligned} \Delta w_j^n - \{ v_m^- [1 - 0.5(1 - v_m^-) (\phi_{m,j+1}^- - \phi_{m,j}^+)] \Delta_{j+1/2} w_m \\ - v_m^+ [1 + 0.5(1 - v_m^+) (\phi_{m,j}^- - \phi_{m,j-1}^+)] \Delta_{j-1/2} w_m \}^n = 0 \end{aligned} \quad (4.3)$$

with $v_m^\pm = \lambda_m^\pm \tau/h$ and

$$\phi(r_{m,j}^{\pm}) = \max(-1, \min(r_{m,j}^{\pm}, 1)) \quad (4.4)$$

where

$$r_{m,j}^{\pm} = \begin{cases} (\Delta_{j-1/2} w_m^n / \Delta_{j+1/2} w_m^n)^{\pm 1} & \text{for } \Delta_{j\pm 1/2} w_m^n \neq 0 \\ 0 & \text{for } \Delta_{j\pm 1/2} w_m^n = 0 \end{cases} \quad (4.5)$$

The nonlinear system case is recovered in two steps [5]: first, the TVDUA scheme in (4.3) is modified as

$$\begin{aligned} \Delta w_{m,j}^n &= \{v_{m,j+1/2}^- [1 - 0.5(1 - v_{m,j+1/2}^-) \phi_{m,j+1}^-] \\ &+ 0.5 v_{m,j-1/2}^- (1 - v_{m,j-1/2}^-) \phi_{m,j}^+ \}^n \Delta_{j+1/2} w_m^n \\ &+ \{v_{m,j-1/2}^+ [1 - 0.5(1 - v_{m,j-1/2}^+) \phi_{m,j-1}^+] \\ &+ 0.5 v_{m,j+1/2}^+ (1 - v_{m,j+1/2}^+) \phi_{m,j}^- \}^n \Delta_{j-1/2} w_m^n = 0 \end{aligned} \quad (4.6)$$

to accommodate the nonlinear scalar conservation law case. The stencil selectors $\phi(r_{m,j}^{\pm})$ remain the same as in (4.4). Following a multiplication of (4.6) by the set of the right eigenvectors r_m^n from the left, a conservative, uniformly second-order accurate high resolution scheme for the nonlinear systems case is easily derived

$$\begin{aligned}
\Delta Q_j^n = & \sum_m \{ r_{m,j+1/2} v_{m,j+1/2}^- [1 - 0.5(1 - v_{m,j+1/2}^-) \phi_{m,j+1}^-] \\
& + 0.5 r_{m,j-1/2} v_{m,j-1/2}^- (1 - v_{m,j-1/2}^-) \phi_{m,j}^+ \}^n \Delta_{j+1/2} w_m^n \\
& + \sum_m \{ r_{m,j-1/2} v_{m,j-1/2}^+ [1 - 0.5(1 - v_{m,j-1/2}^+) \phi_{m,j-1}^+] \\
& + 0.5 r_{m,j+1/2} v_{m,j+1/2}^+ (1 - v_{m,j+1/2}^+) \phi_{m,j}^- \}^n \Delta_{j-1/2} w_m^n = 0 \quad (4.7)
\end{aligned}$$

Although it is no longer necessarily TVD, the scheme in (4.7) gives highly resolved solutions in regions of smooth $Q(x,t)$ while spurious oscillations are suppressed in regions of rapid changes in gradient.

NUMERICAL EXAMPLES

Consider the constant coefficient case

$$q_t + q_x = 0 \quad (5.1a)$$

with

$$q(x,t=0) = \sin \pi x, \quad 0 \leq x \leq 2 \quad (5.1b)$$

and periodic boundary conditions for an initial numerical test. The exact solution to (5.1) for any given time is easily computed by solving the characteristic equation

$$q(x,t) = \sin \pi(x-t) \quad (5.2)$$

Let the interval $[0,2]$ be divided into $(J-1)$ equidistant computational cells with their centroids being defined at

$$x_j = 2(j-1)/(J-1), \quad 1 \leq j \leq J \quad (5.3)$$

Figure 1 shows comparisons of the exact solution (solid line) with three sets of numerical results, at $t=2$ with $\tau/h=0.8$ and $J=21$. As expected, the first-order accurate TVD scheme (fig. 1a) approximates the exact solution rather poorly: a consequence of its hefty inherent numerical dissipation. A comparison of figures 1b and 1c demonstrates how the degeneracy to first-order accuracy at local extrema in the second-order accurate TVD scheme adversely affects the global accuracy in the neighborhood of extrema. This results is substantiated by the grid refinement study in figure 2 for $J=21, 41$, and 81 . The computations with the first-order TVD scheme converge with first-order accuracy in both the L_1 and the L_∞ error norm. The TVDUA scheme is second-order accurate in L_1 as well as in L_∞ ; in between lies the second-order TVD scheme which is second-order accurate in L_1 but only first-order accurate in L_∞ .

Now we turn to examine numerical approximations to a nonlinear scalar problem defined by

$$q_t + (q^2/2)_x = 0 \quad (5.4a)$$

$$q(x,t=0) = \alpha + \beta \sin\pi(x+\gamma), \quad 0 \leq x \leq 2 \quad (5.4b)$$

$$q(x=0, \cdot) = q(x=2, \cdot) \quad (5.4c)$$

For $0 < t < t_0$, the solution to (5.4) is smooth and is "exactly" computed by using, for instance, Newton-Raphson iterations to solve the characteristic relation

$$q(x,t) = \alpha + \beta \sin \pi(x + \gamma - qt), \quad 0 < t < t_0 \quad (5.5)$$

When $t = t_0$ a shock develops at x_0 and moves with its shock speed q_0 for some time before it starts interacting with the expansion wave which brings about a rapid decay of the solution. The values q_0 , t_0 , and x_0 are computed by requiring $\partial q / \partial x \rightarrow \infty$ and $\partial^2 q / \partial x^2 = 0$. We find that a shock develops at $t_0 = 1/\beta\pi$, $x_0 = 1 - \gamma + \alpha/\beta\pi$, and that it moves for some time with its speed $q_0(x_0, t_0) = \alpha$.

Figures 3, 4, and 5 present computations of (5.4) for $\alpha=1$, $\beta=0.5$, and $\gamma=0$ (i.e., $q(x,0) = 1 + 0.5 \sin \pi x$); thus, a shock forms at $t=2/\pi$. The calculations are carried out on meshes as defined in (5.3). In figure 3, the "exact" solution (solid line) is compared with three calculations of increasing accuracy at $t=0.3$ with $\tau/h=0.6$ and $J=21$. Particularly the approximation of the "exact" solution in the region $1.5 < x < 2$ reveals the superior resolution capability of the TVDUA scheme when compared to the first- and second-order accurate TVD schemes. The corresponding grid refinement study in figure 4 shows the convergence of the L_1 and the L_∞ error norms for $J=21, 41$, and 81 . The solutions with the first-order TVD scheme converge with first-order accuracy in both the L_1 and the L_∞ error norm. The second-order TVD scheme proves to be second-order accurate in the L_1 norm, but it is barely more accurate than the first-order TVD scheme in the L_∞ error norm. The TVDUA scheme is clearly second-order accurate in the L_1 norm and

almost second-order accurate in the L_∞ norm. The effect of error accumulation in time becomes apparent when the initial differences between "exact" and numerical solutions for $t=0.3$ in figure 3 are compared with the corresponding results in figure 5 for $t \approx 2/\pi$ (i.e., when the shock develops). The first-order accurate results deviate considerably from the "exact" solution. The second-order TVD scheme captures the forming shock quite decently while the TVDUA scheme "nails" the "exact" solution.

Finally, the TVDUA schemes are put to test when applied to nonlinear hyperbolic systems. It is difficult to find a suitable test case since the extra resolution power of our TVDUA schemes like that of the UNO schemes by Harten and Osher [4] manifests itself best across smooth extrema of a solution. Consequently, we found the commonly chosen shock tube problems of Lax or Sod [2,3,4] not very illustrative because their solutions are just constant states separated by waves. In order to at least achieve a smooth variation of states, we consider a one-dimensional flow in a duct of variable cross-sectional area $B(x)$ which is described by

$$\begin{aligned} Q_t + F_x &= H \\ Q &= (\rho, \rho u, E)^T \\ F &= (\rho u, (\rho u^2 + p), (E + p)u)^T \\ H &= (0, p(d \ln B / dx), 0)^T \end{aligned} \tag{5.6}$$

with the density ρ , the velocity u , the pressure p , and the internal energy per unit volume $E = \rho e + \rho u^2 / 2$ (e : specific internal energy). The eigenvalues of the matrix $A = \partial F / \partial Q$ are

$$\lambda_1 = u-c, \lambda_2 = u, \lambda_3 = u+c \quad (c: \text{ speed of sound}).$$

The corresponding sets of orthonormal left and right eigenvectors are given elsewhere (cf. [2,8,18]). Using Roe's averaging [8] to compute the mean value matrix $\bar{A}(Q_j, Q_{j+1})$, the approximate Riemann solver as described in Section 2 is employed to obtain a first-order upwind approximation to (5.6). The second-order versions of that baseline scheme are constructed by formally extending the TVD and TVDUA concept to hyperbolic systems as described in Section 4.

The geometry of the divergent duct which is depicted in figure 6, is taken from a rather common benchmark test [2,10,11,18]. Specifying

$$\rho_{in} = 0.502, \quad u_{in} = 1.299, \quad e_{in} = 1.897$$

at the inflow cross section, a stationary compression shock forms at $x = 4.816$ [13,18] when $\rho_{out} = 0.776$ at the outflow cross section. The shock causes a problem in quantitatively determining the accuracy of the employed schemes, since the notion of accuracy being strongly coupled with the notion of truncation error in finite-difference methods becomes immaterial for nondifferentiable solutions. Despite this shortcoming, the results in figures 7 and 8 demonstrate the superior resolution power of the TVDUA schemes when compared to the first- and second-order TVD schemes. All computations are carried out on equidistant meshes with just 21 grid points and with $\tau/h=0.2$. The "exact" solution (solid line) and their numerical approximations are plotted in terms of density (fig. 7) and velocity (fig. 8) as a function of x . The first-order TVD scheme does not only smear the shock, but it also

noticeably misses the "exact" solution for $2.5 \leq x \leq 7.5$: a range consisting of ten (!) grid points. The approximate solution computed with the second-order TVD scheme feels the ripples of the shock in a narrower regime with $3 \leq x \leq 6$ which translates to 7 grid points. Both TVD schemes resolve the shock with two intermediate zones, whereas, the TVDUA scheme captures the shock in just one interval. The TVDUA results also follow the "exact" solution much closer ahead and aft of the shock.

CONCLUDING REMARKS

A class of new explicit, uniformly second-order accurate high resolution schemes, which are TVD in the scalar, or linear systems, case, has been constructed, analyzed, and verified in several test calculations. Compared to conventional high resolution schemes which switch back to first-order accuracy at extrema of the solution, the novel schemes here called TVDUA schemes, yield a gain in accuracy at virtually no computational extra cost. The gain in accuracy becomes most apparent when solutions with smooth extrema are approximated as in the test calculations presented for scalar conservation laws. When dealing with essentially discontinuous solutions, the TVDUA schemes produce nonoscillatory solutions, they resolve shocks with just one interval, and they sustain their extra resolution capability almost unaffected even in the neighborhood of a strong singularity. Future work will concentrate on the extension of the novel TVDUA schemes to multi-dimensional problems.

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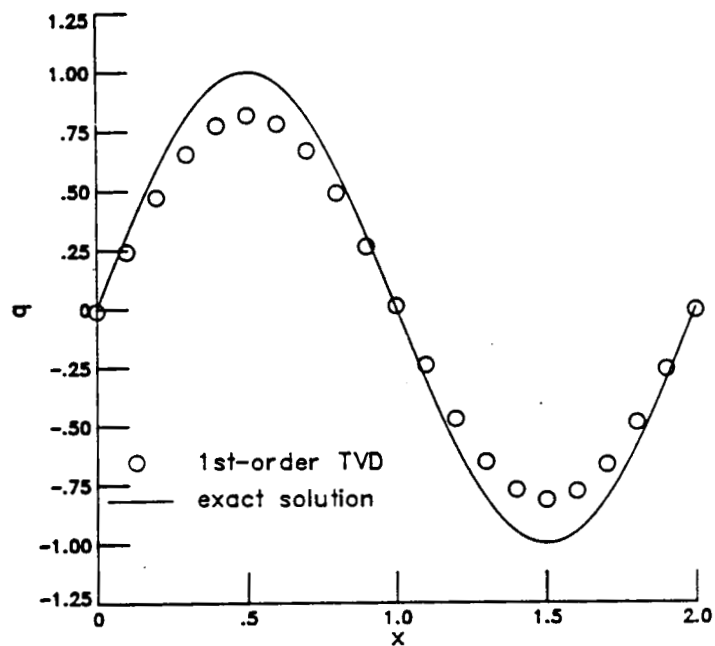


Figure 1(a)

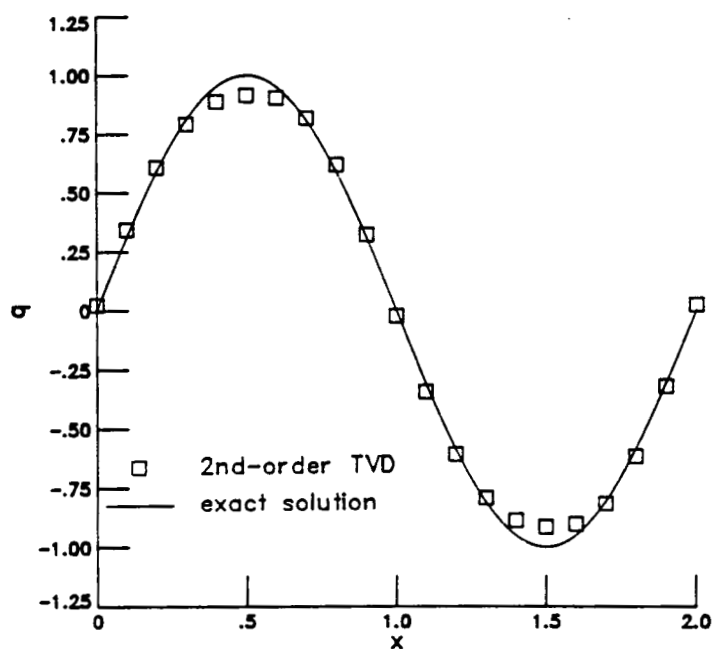


Figure 1(b)

Figure 1. Comparison of exact and numerical solutions for an one-dimensional, linear wave equation problem: $t=2$, $\tau/h=0.8$, 21 grid points.

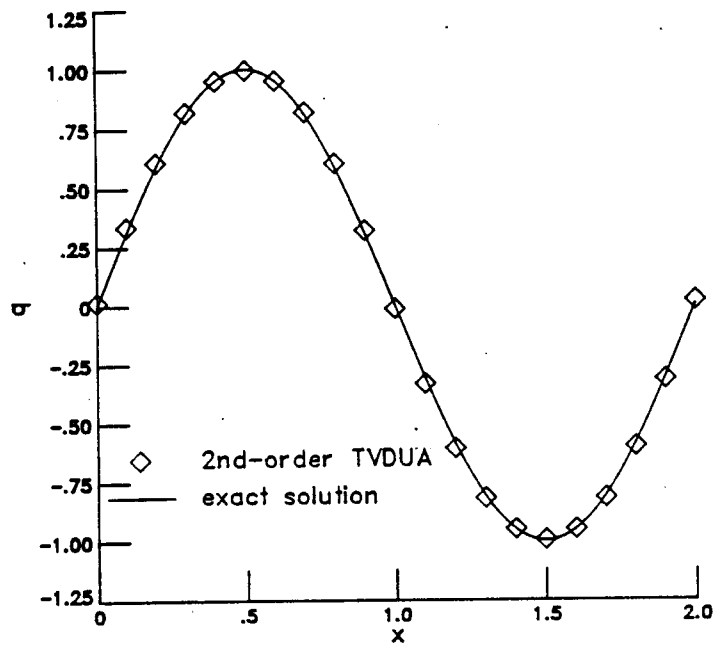


Figure 1(c)

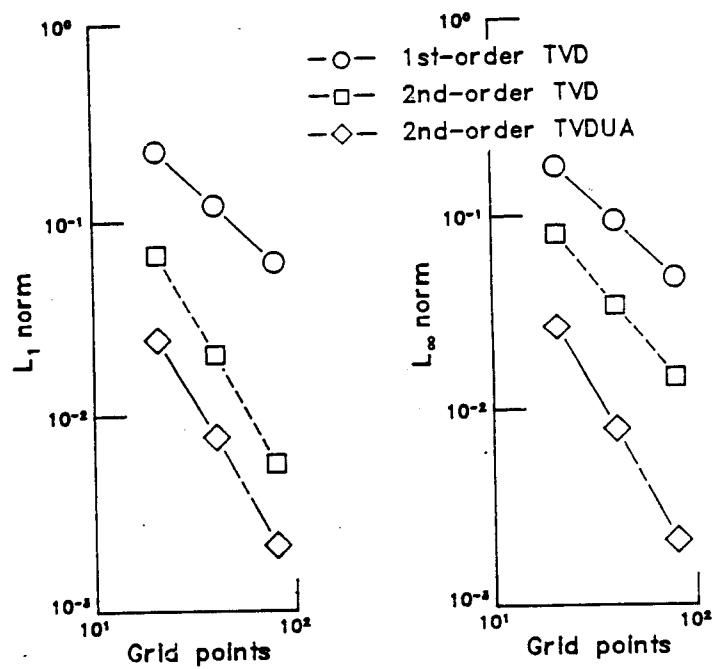


Figure 2. Error norms for numerical solutions to an one-dimensional, linear wave equation: $t=2$, $\tau/h=0.8$.

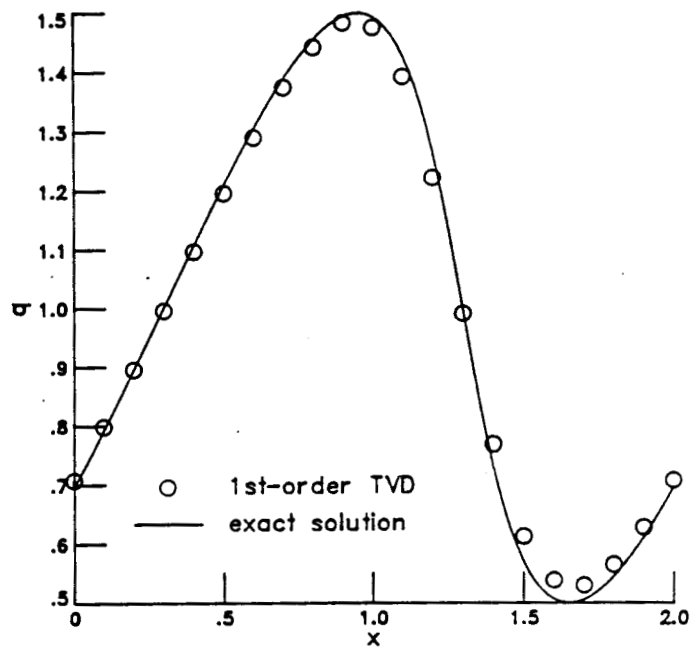


Figure 3(a)

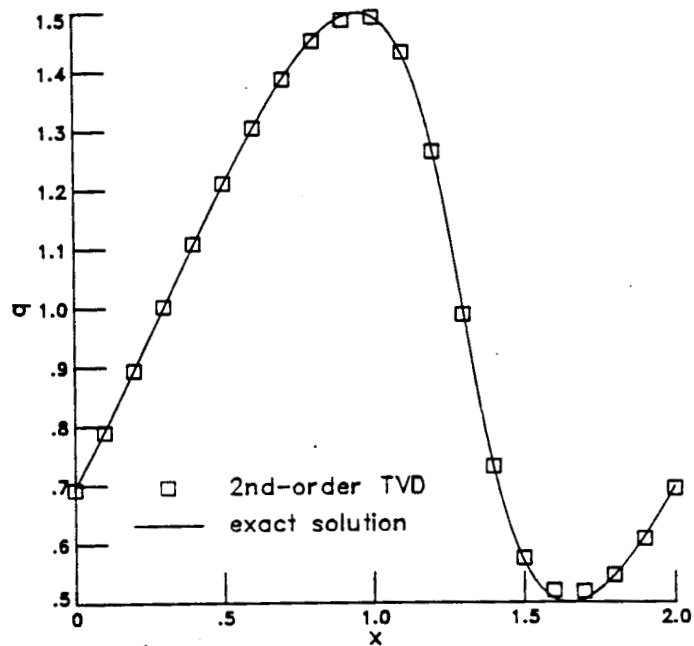


Figure 3(b)

Figure 3. Smooth exact and numerical solutions for an one-dimensional, nonlinear wave equation problem: $t=0.3$, $\tau/h = 0.6$, 21 grid points.

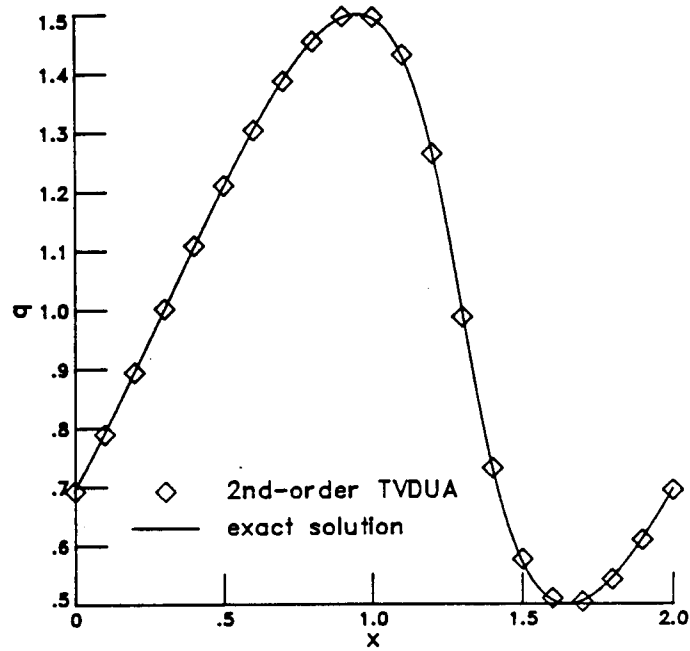


Figure 3(c)

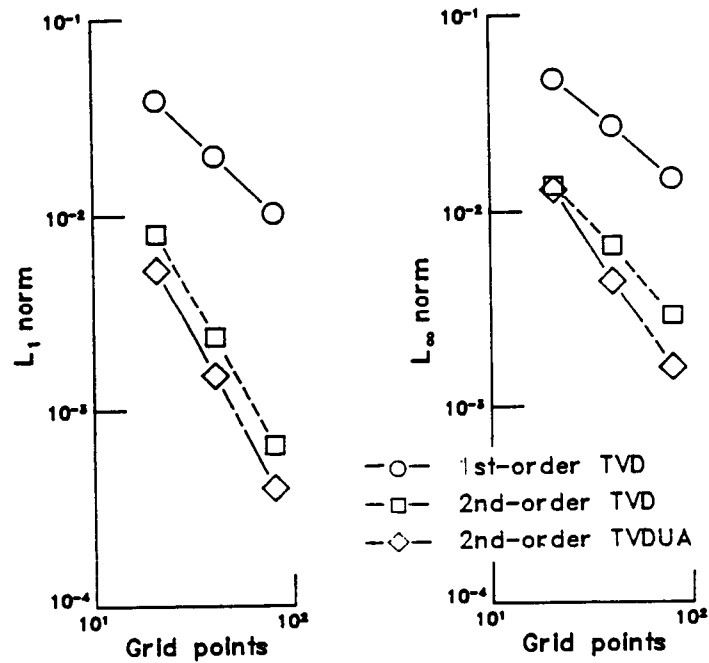


Figure 4. Error norm for smooth numerical solutions to an one-dimensional, nonlinear wave equation: $t=0.3$, $\tau/h=0.6$.

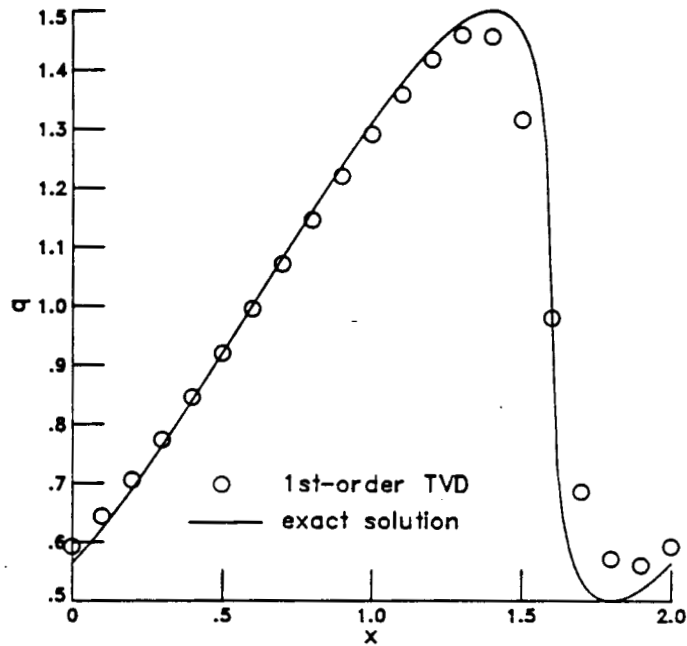


Figure 5(a)

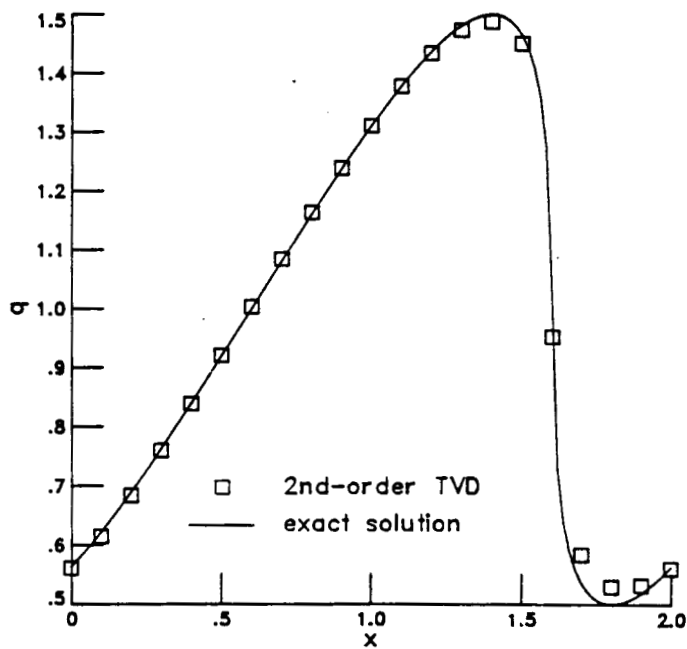


Figure 5(b)

Figure 5. Exact and numerical solutions for an one-dimensional, wave equation problem at the onset of a shock formation: $t \approx 2/\pi$, $\tau/h=0.6$, 21 grid points.

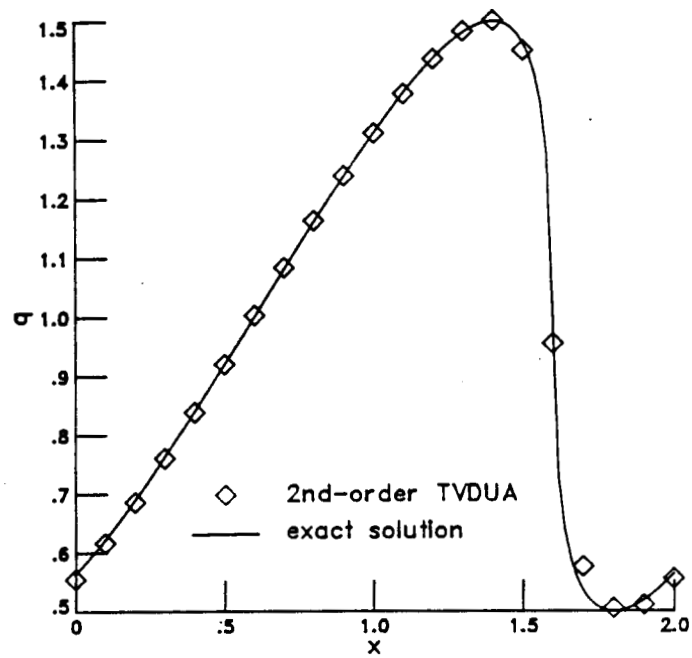


Figure 5(c)

$$A(x)/A^* = 1.398 + 0.347 \tanh(0.8x - 4)$$

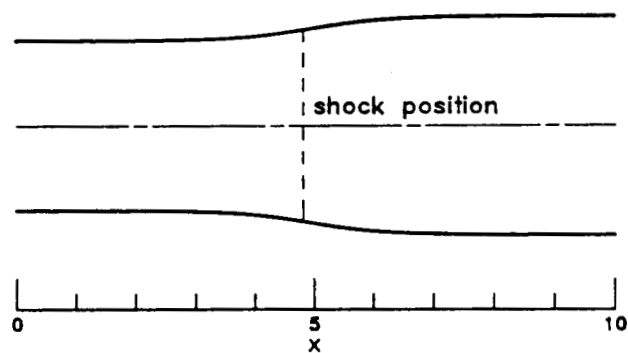


Figure 6. Duct geometry.

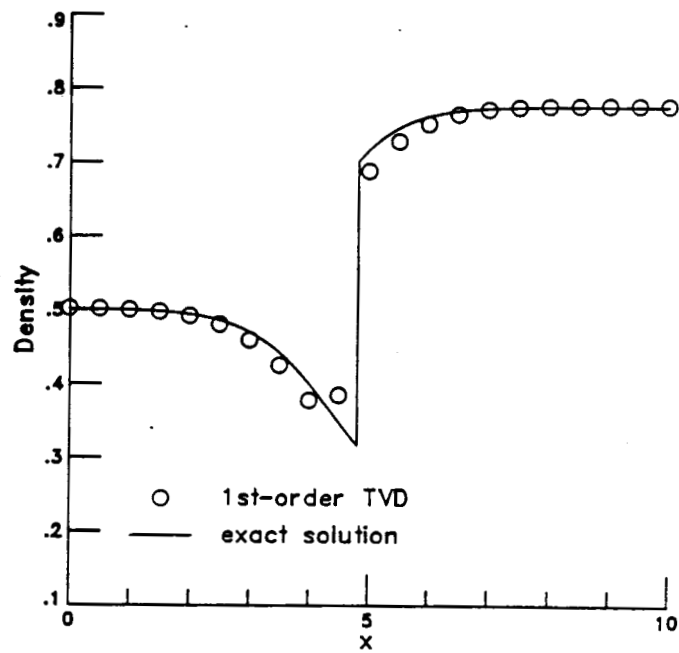


Figure 7(a)

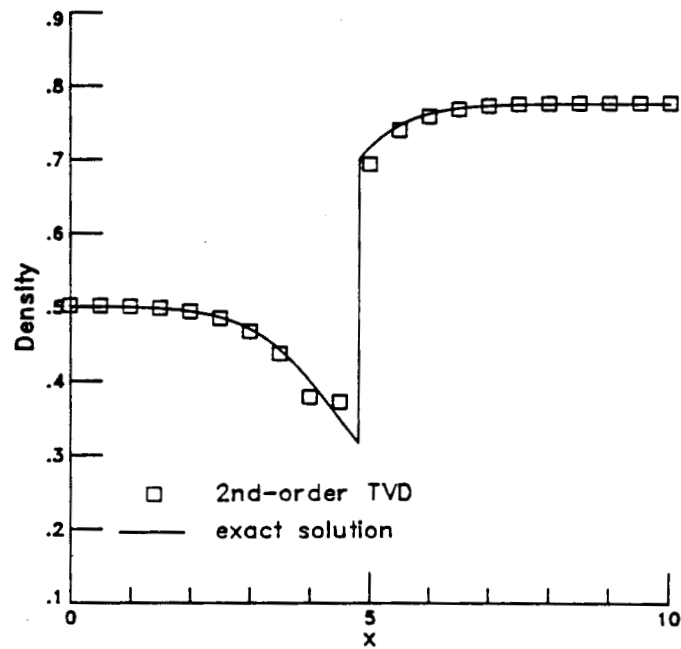


Figure 7(b)

Figure 7. Exact and numerical density solutions for an one-dimensional, divergent nozzle flow: $\tau/h=0.2$, 21 grid points.

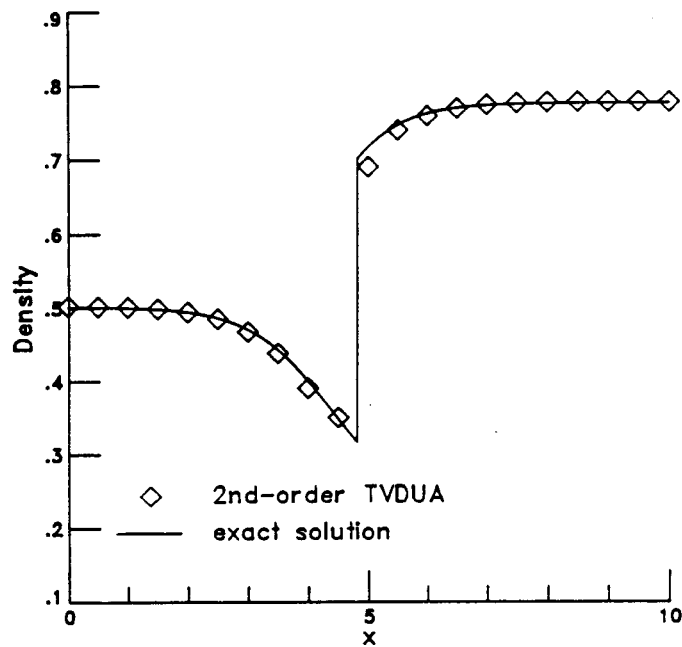


Figure 7(c)

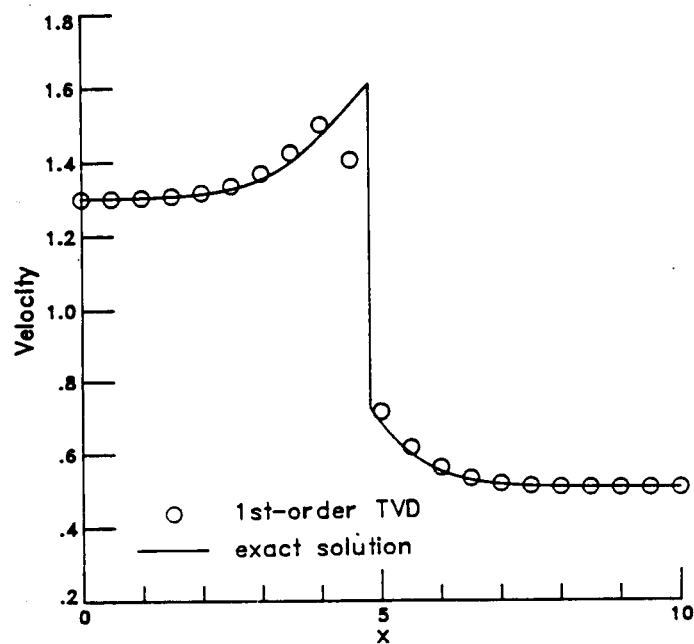


Figure 8(a)

Figure 8. Exact and numerical velocity solutions for an one-dimensional, divergent nozzle flow: $\tau/h=0.2$, 21 grids points.

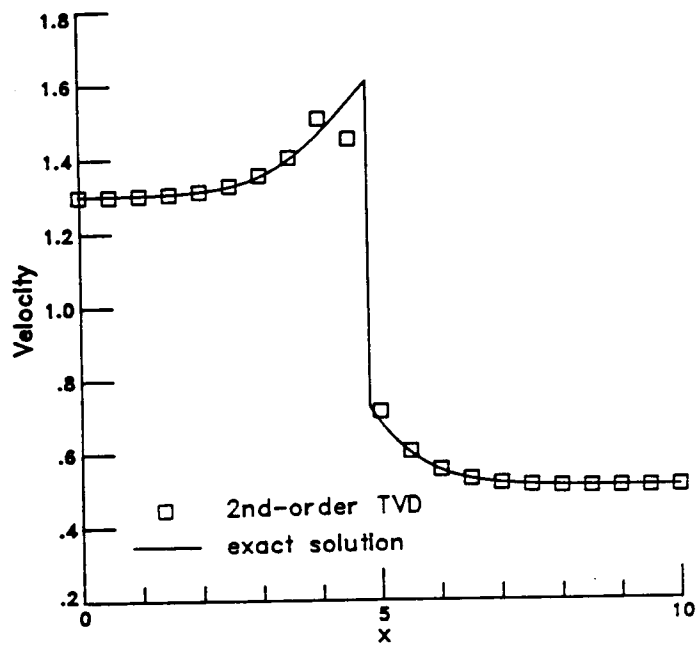


Figure 8(b)

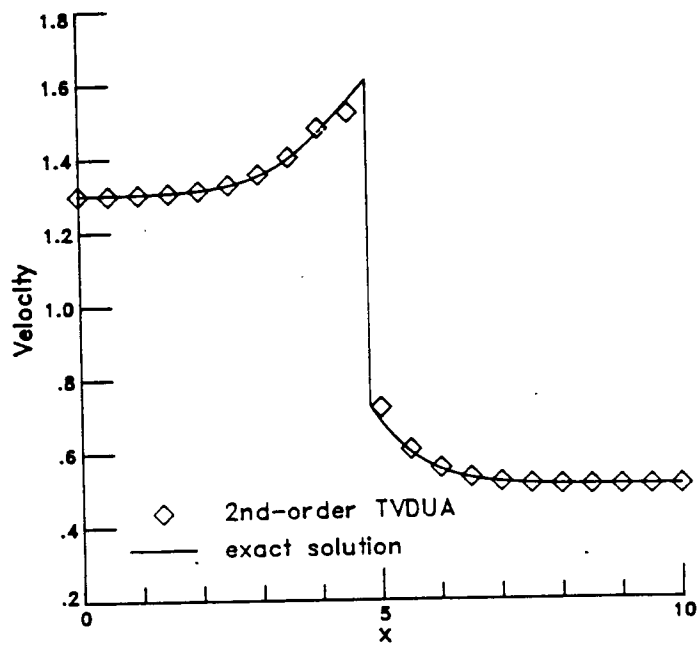


Figure 8(c)

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16. Abstract Explicit second-order accurate finite-difference schemes for the approximation of hyperbolic conservation laws are presented. These schemes are nonlinear even for the constant coefficient case. They are based on first-order accurate upwind schemes. Their accuracy is enhanced by locally replacing the first-order one-sided differences with either second-order one-sided differences or central differences or a blend thereof. The appropriate local difference stencils are selected such that they give TVD schemes of uniform second-order accuracy in the scalar, or linear systems, case. Like conventional TVD schemes, the new schemes avoid a Gibbs phenomenon at discontinuities of the solution, but they do not switch back to first-order accuracy, in the sense of truncation error, at extrema of the solution. The performance of the new schemes is demonstrated in several numerical tests.					
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